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On the Propagation of Confined Waves along the Geodesics

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1. Introduction

It is well known that if a solution u of the wave equation in $\mathbf{R}^n$ is such that $u(0)$ and $u_t(0)$ have their supports in some ball $B(x_0, r)$, then $u(t)$ is supported by $B(x_0, r+|t|)$. The same property holds for the solutions of the wave equation in a bounded domain Ω, with Dirichlet boundary condition, as long as $B(x_0, r+|t|) \subset \Omega$ (see [3, Lemma 2.9]). However, this does not take into account the geometry of the domain Ω.

In the theory of exact controllability, it is essential to understand how the wave equation propagates the information. We show here that the support of the solution propagates with velocity at most 1 along the geodesics of the domain Ω. We apply this result to some specific examples, where we establish in particular the optimality of certain results of Haraux [4] (see also Bardos, Lebeau, and Rauch [1]).

More precisely, let us briefly recall the problem of exact controllability for the wave equation. Let $\Omega \subset \mathbf{R}^n$ be an open set, and let $\omega \subset \Omega$ be a sub-domain. For any $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$, we look for $T > 0$ and $h = h(t, x) \in L^2(0, T, L^2(\Omega))$ such that $\operatorname{supp}(h) \subset [0, T] \times \omega$ and for which the unique solution y of

$$\begin{aligned} y'' - \Delta y &= h, && \text{in } (0, T) \times \Omega; \\ y &= 0, && \text{on } (0, T) \times \partial\Omega; \\ y(0, x) &= y^0(x), && \text{in } \Omega; \\ y'(0, x) &= y^1(x), && \text{in } \Omega \end{aligned} \tag{1.1}$$

satisfies $y(T, x) = y'(T, x) = 0$ in Ω. When the H.U.M. method of J.-L.

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Lions [6–8] is applied, this problem reduces to the study of the homogeneous problem

$$
\begin{aligned}
u'' - \Delta u &= 0, && \text{in} \quad (0, T) \times \Omega; \\
u &= 0, && \text{on} \quad (0, T) \times \partial\Omega; \\
u(0, x) &= \varphi(x), && \text{in } \Omega; \\
u'(0, x) &= \psi(x), && \text{in } \Omega,
\end{aligned} \tag{1.2}
$$

where $(\varphi, \psi) \in H_0^1(\Omega) \times L^2(\Omega)$. More precisely, if we consider the semi-norm p defined on $H_0^1(\Omega) \times L^2(\Omega)$ by

$$p(\varphi, \psi) = \left(\int_0^T \int_\omega |u(t, x)|^2 \, dx \, dt \right)^{1/2},$$

where u is the solution of (1.2), the solution of the exact controllability problem for (1.1) amounts to showing that p is in fact a norm on $H_0^1(\Omega) \times L^2(\Omega)$ and identifying the completion of $H_0^1(\Omega) \times L^2(\Omega)$ for the norm p (see J.-L. Lions [6–8]). Among others, an important question is, What is the minimal value of T (depending in particular on Ω and ω) for which p is a norm?

As in Haraux [4], we introduce the following notation. For every pair (x, y) of points in Ω, we denote by $\delta(x, y)$ the infimum of the lengths of all polygonal lines joining x and y and contained in Ω. We define the "geodesic distance" $d(\omega_1, \omega_2)$ between two subsets ω_1 and ω_2 of Ω by

$$d(\omega_1, \omega_2) = \operatorname{Inf}\{\delta(x, y),\ y \in \omega_1,\ x \in \omega_2\}.$$

In addition, for any subset ω of Ω and for every $x \in \Omega$, we set

$$\delta(x, \omega) = \operatorname{Inf}\{\delta(x, y),\ y \in \omega\}.$$

Moreover we define

$$\delta(\Omega, \omega) = \operatorname{Sup}\{\delta(x, \omega),\ x \in \Omega\}.$$

Finally the "geodesic diameter" of Ω is defined as

$$\delta(\Omega) = \operatorname{Sup}\{\delta(x, y),\ x \in \Omega,\ y \in \Omega\}.$$

It is shown in Haraux [4] that p is a norm on $H_0^1(\Omega) \times L^2(\Omega)$, provided the following holds:

$$T > 2\,\delta(\Omega, \omega). \tag{1.3}$$

We give here some examples of domains in which condition (1.3) is sharp. However, we do not know whether or not condition (1.3) is sharp for a general domain. The paper is organised as follows. In Section 2 we prove a preliminary result, and in Section 3 we give three examples of application to some specific domains.

2. The Model Case

We consider a bounded, open, connected set $\Omega \subset \mathbf{R}^n$, $n \geqslant 2$. We assume that $\Omega = \Omega' \cup K \cup \Omega''$, where Ω' and Ω'' are two connected, open sets, $\Omega' \cap \Omega'' = \Omega' \cap K = \Omega'' \cap K = \varnothing$, and K is of the form

$$K = \bigcup_{s \in [a,b]} \omega_s,$$

with $-\infty < a < b < +\infty$ and $\omega_s = \{s\} \times \delta_s$, where $(\delta_s)_{s \in [a,b]}$ is a family of connected, open subsets of $\mathbf{R}^{n-1}$ (see Fig. 1).

We assume that K is smooth in the sense that there exist two open sets D' and D'' such that $D'' \cup K \cup D'$ is a connected open set, with a boundary of class C^2 (see Fig. 2).

For $\sigma \in [a, b]$, we set (see Fig. 3)

$$\Omega_\sigma = \Omega' \cup K_\sigma, \qquad \text{where} \quad K_\sigma = \bigcup_{s \in (\sigma,b]} \omega_s.$$

Ω_σ is also a connected open subset of $\mathbf{R}^n$. We consider the boundary $\partial\Omega_\sigma$ of Ω_σ, which contains ω_σ, and we set

$$\Gamma_\sigma = \partial\Omega_\sigma \backslash \omega_\sigma.$$

Consider a positive number T, such that

$$T \leqslant b - a.$$

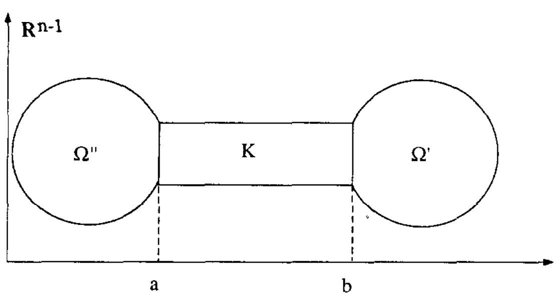

FIGURE 1

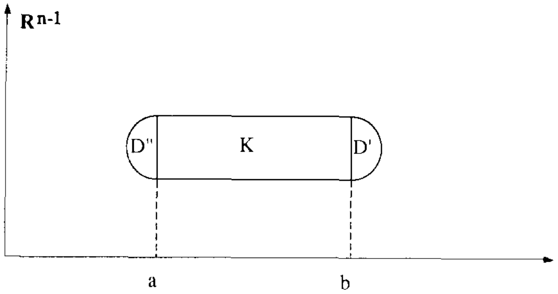

FIGURE 2

Let $V \in L^1(0, T, L^\gamma(\Omega))$, with $\gamma \geqslant n$ and $\gamma > 2$ if $n = 2$, and let $u \in C([0, T], H_0^1(\Omega)) \cap C^1([0, T], L^2(\Omega))$ solve

$$u_{tt} - \Delta u + u = Vu, \qquad \text{in } \mathscr{D}'((0, T) \times \Omega). \tag{2.1}$$

Observe that by Sobolev's inequality, $u \in C([0, T], L^{2\gamma/(\gamma-2)}(\Omega))$; and so $Vu \in L^1(0, T, L^2(\Omega))$. Thus, (2.1) makes sense.

Our main result of this section is the following.

PROPOSITION 2.1. *Assume* $u(0, \cdot) = u_t(0, \cdot) = 0$ *a.e. in* $\Omega_a = \Omega' \cup K$. *Then* $u(\sigma, \cdot) = 0$ *a.e. in* $\Omega_{a+\sigma}$, *for every* $\sigma \in [0, T]$.

Proof. First, observe that we can replace Ω' by any other connected, open set U' such that $U = \Omega'' \cup K \cup U'$ is a connected, open set. To see this, assume Proposition 2.1 holds in Ω, and let $v \in C([0, T], H_0^1(U)) \cap C^1([0, T], L^2(U))$ be as in the statement of Proposition 2.1, with Ω replaced by U. Let $(\varphi, \psi) = (v(0), v_t(0))$, and let ζ and ξ be defined by

$$\zeta = \begin{cases} \varphi, & \text{on } \Omega'' \cup K; \\ 0, & \text{on } \Omega'; \end{cases} \qquad \xi = \begin{cases} \psi, & \text{on } \Omega'' \cup K; \\ 0, & \text{on } \Omega'. \end{cases}$$

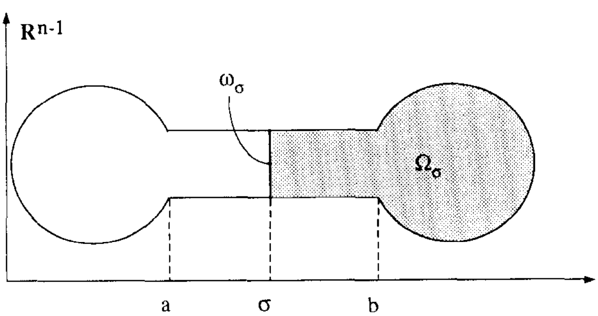

FIGURE 3

Since φ is supported in Ω'', we have $(\zeta, \xi) \in H_0^1(\Omega) \times L^2(\Omega)$. Let $u \in C([0, T], H_0^1(\Omega)) \cap C^1([0, T], L^2(\Omega))$ be the solution of (2.1) such that $(u(0), u_t(0)) = (\zeta, \xi)$, with V replaced by the function W equal to V in $\Omega \cap U$, and to 0 outside (see [5] for the existence and uniqueness of u). It is clear that u satisfies the hypotheses of Proposition 2.1. In particular, the function $\hat{u}$ defined for $t \in [0, T]$ by

$$\hat{u}(t) = \begin{cases} u(t), & \text{in } \quad \Omega'' \cup K; \\ 0, & \text{in } D' \end{cases}$$

is such that $\hat{u} \in C([0, T], H_0^1(U)) \cap C^1([0, T], L^2(U))$, and $\hat{u}$ solves the same equation as v does. By uniqueness, we have $\hat{u} = v$; and so v verifies the conclusions of Proposition 2.1.

Therefore, in all the sequel, we assume that $\Omega' = D'$. For $\sigma \in [0, T]$, we let

$$I(\sigma) = \int_{\Omega_{a+\sigma}} \{u_t^2 + |\nabla u|^2 + u^2\} \, dx. \tag{2.2}$$

We have $I(0) = 0$; and so Proposition 2.1 is a consequence of the following lemma.

LEMMA 2.2. *Let* $u \in C([0, T], H_0^1(\Omega)) \cap C^1([0, T], L^2(\Omega))$ *solve* (2.1), *and let I be defined by* (2.2). *Then there exists K, depending on V, such that*

$$I(\sigma) \leqslant I(0) \exp(K\sigma), \qquad \textit{for every} \quad \sigma \in [0, T]. \tag{2.3}$$

Proof. The solution $u \in C([0, T], H_0^1(\Omega)) \cap C^1([0, T], L^2(\Omega))$ of (2.1) such that $u(0) = f$ and $u_t(0) = g$ depends continuously on $(f, g) \in H_0^1(\Omega) \times L^2(\Omega)$ and on $V \in L^1(0, T, L^\gamma(\Omega))$ (see [5]). Note also that the integral in (2.2) depends continuously on u in $H_0^1(\Omega)$ and on u_t in $L^2(\Omega)$. Thus, we can assume that $u(0)$ and $u_t(0)$ are functions of $\mathscr{D}(\Omega)$ and that $V \in C([0, T], \mathscr{D}(\Omega))$, so that in particular, we have $\Delta u \in C([0, T], L^2(\Omega))$ (see [5]). Since by assumption $D'' \cup K \cup \Omega'$ is smooth, it follows easily that $u \in C([0, T], H^2(\Omega_{a+\sigma}))$, for every $\sigma \in (0, T)$ (multiply u by any smooth function supported in Ω_a and equal to 1 on $\Omega_{a+\sigma}$, and apply the standard regularity theorem in $D'' \cup K \cup \Omega'$; see, for example, [2, Theorem IX.25]). Therefore, the calculations below make sense. We have

$$I(\sigma) = \int_{\Omega'} \{u_t^2 + |\nabla u|^2 + u^2\} \, dx + \int_{a+\sigma}^{b} ds \int_{\omega_s} \{u_t^2 + |\nabla u|^2 + u^2\} \, d\rho,$$

$$\text{for} \quad \sigma \in [0, T].$$

Therefore,

$$\frac{dI}{d\sigma}=\int_{\Omega_{a+\sigma}}\frac{\partial}{\partial t}\{u_t{}^2+|\nabla u|^2+u^2\}\,dx-\int_{\omega_{a+\sigma}}\{u_t{}^2+|\nabla u|^2+u^2\}\,d\rho. \quad (2.4)$$

Now, observe that by (2.1),

$$\frac{\partial}{\partial t}\{u_t{}^2+|\nabla u|^2+u^2\}=2Vuu_t+2\nabla\cdot(u_t\cdot\nabla u). \quad (2.5)$$

It follows from (2.4), (2.5), and Green's formula that

$$\frac{dI}{d\sigma}=2\int_{\Omega_{a+\sigma}}Vuu_t\,dx+2\int_{\partial\Omega_{a+\sigma}}(u_t\cdot\nabla u)\cdot\mathbf{n}\,d\rho$$
$$-\int_{\omega_{a+\sigma}}\{u_t{}^2+|\nabla u|^2+u^2\}\,d\rho=A_1+A_2+A_3. \quad (2.6)$$

Next, let $\varphi(\sigma)=\|V(\sigma)\|_{L^\gamma}$. Observe that from Hölder's and Sobolev's inequalities, it follows that

$$A_1\leqslant 2\,\|V(\sigma)\|_{L^\gamma}\,\|u(\sigma)\|_{L^{2\gamma/(\gamma-2)}}\,\|u_t(\sigma)\|_{L^2}\leqslant C\varphi(\sigma)\,\|\nabla u(\sigma)\|_{L^2}\,\|u_t(\sigma)\|_{L^2};$$

and so

$$A_1\leqslant C\varphi(\sigma)\,I(\sigma). \quad (2.7)$$

Now, observe that $\partial\Omega_{a+\sigma}=\omega_{a+\sigma}\cup\Gamma_{a+\sigma}$, and that $u_t=0$ on $\Gamma_{a+\sigma}$, so that

$$A_2=2\int_{\omega_{a+\sigma}}(u_t\cdot\nabla u)\cdot\mathbf{n}\,d\rho. \quad (2.8)$$

Note also that

$$|2(u_t\cdot\nabla u)|\leqslant u_t{}^2+|\nabla u|^2.$$

Therefore, (2.6) and (2.8) imply

$$A_2+A_3\leqslant -\int_{\omega_{a+\sigma}}u^2\,d\rho\leqslant 0. \quad (2.9)$$

It follows from (2.6), (2.7), and (2.9) that

$$I'(\sigma)\leqslant C\varphi(\sigma)\,I(\sigma),\qquad \text{for almost all}\quad \sigma\in(0,T). \quad (2.10)$$

Integrating (2.10), we obtain (2.3) with $K=C\,\|V\|_{L^1(0,T,L^\gamma)}$.

Remark 2.3. Observe that Proposition 2.1 applies to some nonlinear equations. Indeed, let us consider, for example, the equation

$$u_{tt} - \Delta u + g(t, x, u) = 0, \qquad \text{in } \mathscr{D}'((0, T) \times \Omega). \tag{2.11}$$

Assume that

$$|g(t, x, z)| \leqslant W(t, x)\,|z| + C\,|z|^{\rho}, \qquad \text{for every} \quad (t, x, z) \in (0, T) \times \Omega \times \mathbf{R},$$

with $W \in L^1(0, T, L^{\gamma}(\Omega))$ and $(n-2)\rho < n$. Consider a solution $u \in C([0, T], H_0^1(\Omega))$ of (2.11). Then u solves (2.1), with

$$V(t, x) = \frac{g(t, x, u(t, x))}{u(t, x)} \in L^1(0, T, L^{\gamma}(\Omega)).$$

3. Applications

In this section, we give examples of application of Proposition 2.1. In these examples, it is observed that condition (1.3) is sharp. These remarks also apply to show the optimality of certain results of Bardos, Lebeau, and Rauch [1], at least in the examples considered below.

Example 1. Let n be an even integer, and $0 < \varepsilon < \frac{1}{2}$. We consider the open set $\Omega \subset \mathbf{R}^2$ as in Fig. 4, that is

$$\Omega = \{(x, y) \in (0, 1)^2,\ (x, y) \notin E_1 \cup E_2\},$$

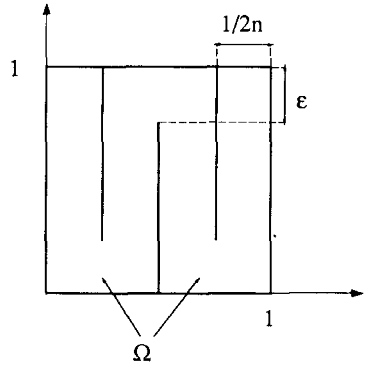

Figure 4

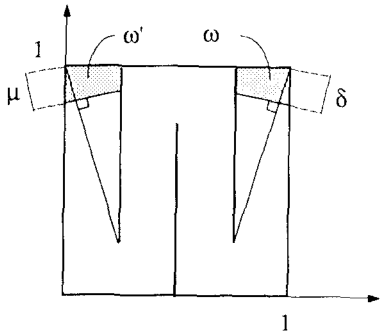

FIGURE 5

where

$$E_1 = \bigcup_{k=1}^{n-1} \{(x, y),\ x = k/n,\ 0 \leqslant y \leqslant 1 - \varepsilon\},$$

$$E_2 = \bigcup_{k=1}^{n} \{(x, y),\ x = k/n - \frac{1}{2n},\ \varepsilon \leqslant y \leqslant 1\}.$$

Let $\delta, \mu > 0$ such that

$$\delta, \mu < \sqrt{(1-\varepsilon)^2 + \frac{1}{4n^2}},$$

and consider the open sets ω and ω' as indicated in Fig. 5. Observe that

$$\delta(\Omega, \omega) = 2\sqrt{(1-\varepsilon)^2 + \frac{1}{4n^2}} + 2(n-1)\sqrt{(1-2\varepsilon)^2 + \frac{1}{4n^2}} - \delta.$$

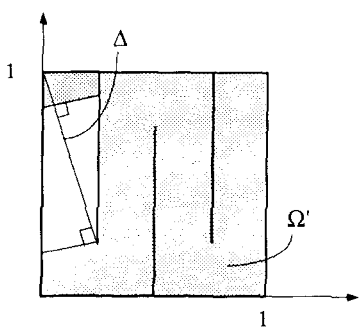

FIGURE 6

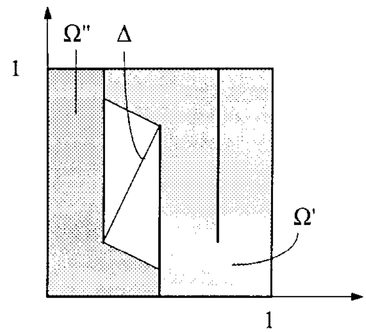

FIGURE 7

Consider $\varphi \in \mathscr{D}(\Omega)$, supported in ω', and let $u \in C(\mathbf{R}, H_0^1(\Omega)) \cap C^1(\mathbf{R}, L^2(\Omega))$ be the solution of

$$u_{tt} - \Delta u = 0, \qquad \text{in } \mathscr{D}'(\mathbf{R} \times \Omega),\ u(0) = \varphi,\ u_t(0) = 0. \tag{3.1}$$

We show that $u \equiv 0$ in ω, for $t \in [-\delta(\Omega, \omega) + \mu, \delta(\Omega, \omega) - \mu]$. We first write $\Omega = \Omega' \cup K \cup \Omega''$, with $\Omega'' = \omega'$ and Ω' as indicated in Fig. 6. We make a change of coordinates, so that the line Δ indicated in Fig. 6 becomes the x-axis. When proposition 2.1 is applied, it follows that $u \equiv 0$ in Ω' (hence in ω), for $t \in [0, h - \mu]$, with

$$h = \sqrt{(1-\varepsilon)^2 + \frac{1}{4n^2}}.$$

Next, we write $\Omega = \Omega' \cup K \cup \Omega''$, with Ω'' and Ω' as indicated in Fig. 7. We make a change of coordinates, so that the line Δ indicated in Fig. 7

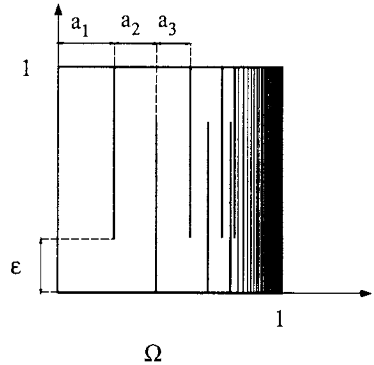

FIGURE 8

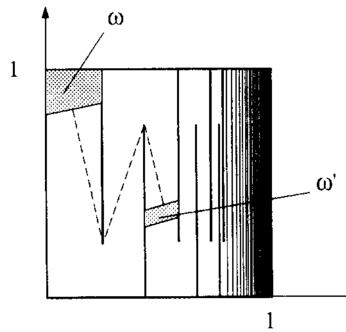

FIGURE 9

becomes the x-axis. When Proposition 2.1 is applied, it follows that $u \equiv 0$ in Ω' (hence in ω), for $t \in [h-\mu, h+\sigma-\mu]$, with

$$\sigma = \sqrt{(1-2\varepsilon)^2 + \frac{1}{4n^2}}.$$

Therefore, $u \equiv 0$ in Ω' (hence in ω), for $t \in [0, h+\sigma-\mu]$. When this argument is iterated, it follows that $u \equiv 0$ in ω, for $t \in [0, \delta(\Omega, \omega) - \mu]$. Changing t to $-t$, we obtain as well that $u \equiv 0$ in ω, for $t \in [-\delta(\Omega, \omega) + \mu, 0]$. This proves the desired result.

Since $\mu > 0$ is arbitrary, it follows that for every $\tau < \delta(\Omega, \omega)$, there exists a solution u of (3.1) that vanishes identically in ω for $t \in [-\tau, \tau]$. Evidently, if we choose $\varphi \neq 0$, we have $u \neq 0$. Therefore, if we choose any $T < 2\delta(\Omega, \omega)$, then p does not define a norm on $H_0^1(\Omega) \times L^2(\Omega)$; and so the lower bound in (1.3) is optimal.

EXAMPLE 2. Let $(a_n)_{n \geqslant 1}$ be a sequence of positive numbers such that $\sum a_n = 1$, and $0 < \varepsilon < \frac{1}{2}$. We consider the open set $\Omega \subset \mathbf{R}^2$ as in Fig. 8, that is,

$$\Omega = \{(x, y) \in (0, 1)^2, (x, y) \notin E_1 \cup E_2\},$$

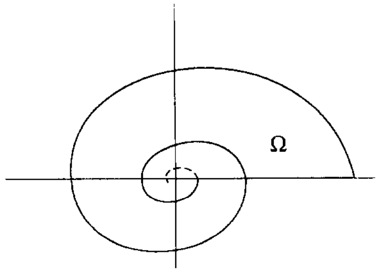

FIGURE 10

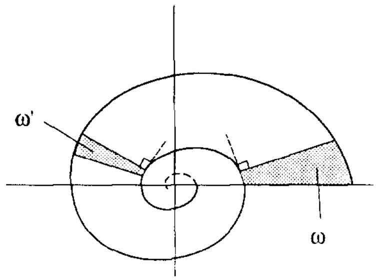

FIGURE 11

where

$$E_1 = \bigcup_{k \geq 1} \left\{ (x, y),\ x = \sum_{m=1}^{2k} a_m,\ 0 \leqslant y \leqslant 1 - \varepsilon \right\},$$

$$E_2 = \bigcup_{k \geq 1} \left\{ (x, y),\ x = \sum_{m=1}^{2k-1} a_m,\ \varepsilon \leqslant y \leqslant 1 \right\}.$$

Here, the geodesic diameter of Ω is $\delta(\Omega) = \infty$. Let $T > 0$, and consider two open subsets ω and ω' of Ω such that the geodesic distance between ω and ω' is T, and located as in Fig. 9 (T is the length of the discontinuous line). Let $\varphi \in \mathscr{D}(\Omega)$, supported in ω', and let $u \in C(\mathbf{R}, H_0^1(\Omega)) \cap C^1(\mathbf{R}, L^2(\Omega))$ be the solution of (3.1). Then, using a method similar to that described for Example 1, it follows that $u(t) = 0$ a.e. in ω, for every $t \in [-T, T]$. Since $T > 0$ is arbitrary, it follows that for any $T > 0$, p does not define a norm on $H_0^1(\Omega) \times L^2(\Omega)$.

EXAMPLE 3. We now consider the case of a spiral in $\mathbf{R}^2$. Let f: $[0, \infty) \to (0, \infty)$ be a smooth decreasing function, and let $\Omega \subset \mathbf{R}^2$ be the open set defined by (see Fig. 10)

$$\Omega = \{\rho e^{i\theta},\ \theta > 0,\ \rho \in (f(\theta + 2\pi), f(\theta))\}.$$

Let ω and ω' be two open subsets of Ω as indicated in Fig. 11.

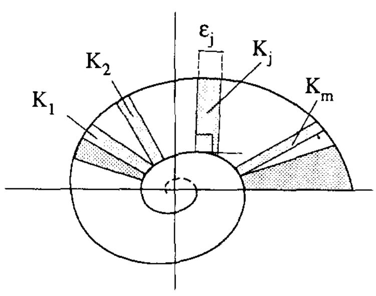

FIGURE 12

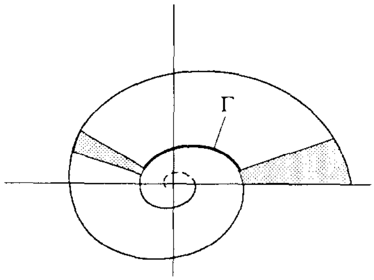

FIGURE 13

Let $\varphi \in \mathscr{D}(\Omega)$, supported in ω', and let $u \in C(\mathbf{R}, H_0^1(\Omega)) \cap C^1(\mathbf{R}, L^2(\Omega))$ be the solution of (3.1). Then, applying m times Proposition 2.1, with the sets $K_j(m)$ as indicated in Fig. 12, it follows that $u(t) = 0$ a.e. in ω, for every $t \in [-T(m), T(m)]$, where

$$T(m) = \varepsilon_1(m) + \cdots + \varepsilon_m(m).$$

If we let $m \to \infty$, with $\varepsilon_j(m) \to 0$, it follows that $T(m)$ converges to the length of the segment Γ of the spiral indicated in Fig. 13, which is the geodesic distance between ω and ω'.

Therefore and as in Example 2, if the spiral has an infinite length it is possible to construct for arbitrarily large values of T a nontrivial solution $u \in C(\mathbf{R}, H_0^1(\Omega)) \cap C^1(\mathbf{R}, L^2(\Omega))$ of (3.1) such that $u(t) = 0$ a.e. in ω, for every $t \in [-T, T]$. It follows that for any $T > 0$, p does not define a norm on $H_0^1(\Omega) \times L^2(\Omega)$.

On the other hand, if the spiral has a finite length, then we can choose ω' such that the geodesic distance between ω and ω' is arbitrarily close to $\delta(\Omega, \omega)$. This shows in this example also the optimality of the lower bound in (1.3).

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